

# Electromagnetically-Induced Frame-Dragging around Astrophysical Objects

Andrés F. Gutiérrez-Ruiz<sup>\*</sup> and Leonardo A. Pachón<sup>†</sup>

*Grupo de Física Atómica y Molecular, Instituto de Física, Facultad de Ciencias Exactas y Naturales,  
Universidad de Antioquia UdeA; Calle 70 No. 52-21, Medellín, Colombia.*

(Dated: June 23, 2015)

Frame dragging (Lense-Thirring effect) is generally associated with rotating astrophysical objects. However, it can also be generated by electromagnetic fields if electric and magnetic fields are simultaneously present. In most models of astrophysical objects, macroscopic charge neutrality is assumed and the entire electromagnetic field is characterized in terms of a magnetic dipole component. Hence, the purely electromagnetic contribution to the frame dragging vanishes. However, strange stars may possess independent electric dipole and neutron stars independent electric quadrupole moments that may lead to the presence of purely electromagnetic contributions to the frame dragging. Moreover, recent observations have shown that in stars with strong electromagnetic fields, the magnetic quadrupole may have a significant contribution to the dynamics of stellar processes. As an attempt to characterize and quantify the effect of electromagnetic frame-dragging in these kind of astrophysical objects, an analytic solution to the Einstein-Maxwell equations is constructed here on the basis that the electromagnetic field is generated by the combination of arbitrary magnetic and electric dipoles plus arbitrary magnetic and electric quadrupole moments. The effect of each multipole contribution on the vorticity scalar and the Poynting vector is described in detail. Corrections on important quantities such the innermost stable circular orbit (ISCO) and the epicyclic frequencies are also considered.

## I. INTRODUCTION

Frame dragging (Lense-Thirring effect) is the quintessential hallmark of general relativity and is the result of the spacetime vorticity. It was detected by the Gravity Probe B [1] and traditionally, it has been associated with rotating stellar astrophysical objects and in other astrophysical contexts, such as Galactic models, the frame dragging has been associated to the presence of magnetogravitational monopoles [2, 3]. Surprisingly, if the astrophysical object does not rotate but possesses both electric and magnetic fields, the spacetime vorticity does not vanish [4–6]. In this case, frame dragging is of purely electromagnetic nature and it is associated with the existence of a nonvanishing electromagnetic Poynting vector around the source [5–9].

In most of the early models of astrophysical objects, macroscopic charge neutrality is assumed [10] and the magnetic field is characterized in terms of a pure dipole component [11–14]. This idea was endorsed by the confirmation of many predicted features of a star with a dipole field using two- and three-dimensional magnetohydrodynamic numerical simulations of magnetospheric accretion [15–18]. Thus, under this configuration the *purely* electromagnetic contribution to the vorticity tensor vanishes. However, astrophysical objects such as strange stars may possess independent electric dipole (see Sec. 10.4 in Ref. [19]) and neutron stars independent electric quadrupole moments (see below), which may lead to the presence of purely electromagnetic contributions to frame dragging. A precise account of those contribu-

tions is not available because of the lack of an analytic exact solution with such a complex electromagnetic field configuration.

Moreover, recent observations have shown that in stars with strong electromagnetic fields, the magnetic quadrupole may have significant contributions on the dynamics of stellar processes. Therefore, it is well known by now that the actual configuration of the magnetic field of strongly magnetized stars may depart from the dipole configuration [20]. As an attempt to describe the spacetime geometry surrounding these kind of astrophysical objects, an analytic solution to the Einstein-Maxwell field equations is constructed here on the basis that the electromagnetic field is generated by the combination of arbitrary magnetic and electric dipole plus arbitrary magnetic and electric quadrupole moments. This analytic exact solution allows for analyzing, e.g., the effect of each multipole contribution on the vorticity scalar and the Poynting vector (see Sec. V). Moreover, it is possible to predict corrections to important quantities such as the innermost stable circular orbit and the epicyclic frequencies.

To motivate further the derivation of the model considered here, the observational evidence for the existence of nondipolar fields in a variety of astrophysical objects is discussed next.

## II. OBSERVATIONAL EVIDENCE OF NON-DIPOLAR FIELDS

Measurements of magnetic fields of strongly magnetized stars, based on the Zeeman–Doppler imaging technique [21], have shown that for these kind of astrophysical objects the magnetic field has a complicated multipolar topology in the vicinity of the star [22–26]. This feature

<sup>\*</sup> [afelipe.gutierrez@udea.edu.co](mailto:afelipe.gutierrez@udea.edu.co)

<sup>†</sup> [leonardo.pachon@udea.edu.co](mailto:leonardo.pachon@udea.edu.co)

certainly is of prime relevance in, e.g., the accretion-disk dynamics in binary systems because if the quadrupole component dominates, then the flow of matter into the star will certainly differ from the well-known dynamics induced by a pure dipole field [25].

Complex configurations of magnetic fields are also present in stars such as the T-Tauri stars. They are young stellar objects of low mass that present variations in their luminosity. An important subclass of this kind of stars are the so-called classical T-Tauri stars (cTTS) because they present accretion from the circumstellar disk [27]. Recent evidence points out that in classical T-Tauri stars, the magnetic field near the star is strongly non-dipolar [26]. In the particular case of V2129 Oph, there exists a dominant octupole, 0.12 T, and a weak dipole component, 0.035 T, of the magnetic field [28]. Understanding the circumstellar disk dynamics, under complex field topologies, could provide insight into the formation of planets and the evolution of the star itself.

The discussion above also applies to white dwarfs. The first observations indicated that only a small fraction of white dwarfs appear to exhibit magnetic fields. However, the observational situation changed significantly by the discovery of strong-field magnetic white dwarfs [29–31], which are known to cover a wide range of field strengths  $\sim 1 - 100$  T and deviates from the simple dipolar configuration [32, 33]. Recent spectropolarimetric observations in white dwarf have shown that, in addition to the dipole term, the quadrupole and octupole terms make significant contributions to the field when it is represented as an axisymmetric multipolar expansion [34]. Many studied cases indicate that higher multipole components or non axisymmetric components may be required in a realistic model of white dwarfs (see, e.g., Ref [35]).

Magnetars constitute an additional source of motivation. They are characterized by their extremely powerful magnetic fields, covering strengths from  $\sim 10^8$  to  $\sim 10^{11}$  T [36, 37], so that they can have occasional violent bursts. However, certain magnetars, such as the SGR 0418+5729, undergo this bursting phenomenon even with a weak magnetic fields ( $\sim 10^8$  T) [38]. For SGR 0418+5729, the observed X-ray spectra cannot be fit with this low field strength; hence, it has been suggested that a hidden non-dipole field component must be present to explain the bursting episodes [39].

In summary, there is sufficient observational evidence to develop a consistent analytic closed representation of the exterior spacetime around stars with non-dipolar magnetic fields.

### III. ANALYTIC FORMULAE OF THE MODEL

By combining the facts that (i) almost all the analytic closed form models for relevant astrophysical objects have been conceived in the frame of stationary axisymmetry geometry (see [40–45] for the case of neutron stars), (ii) powerful tools to construct exact solution to the Einstein-

Maxwell field equation have been developed, e.g. [46–48], and (iii) systematic studies on the construction of exact solution from its physical content have been performed [49, 50], a new analytic exact solution to the Einstein-Maxwell field equations is introduced below. This solution provides physical insight, e.g., into the influence of high order electromagnetic multipole moments in the frame dragging and in studying quasi-periodic oscillations (QPOs), which become a useful tool to identify the characteristics of the compact objects present in Low Mass X-Ray Binaries (LXRB) [51–53]. The model presented here is a member of the  $N$ -solitonic solution derived in Ref. [54]. In Appendix A the relevant equations of the derived metric are summarized.

In terms of the quasicylindrical Weyl-Lewis-Papapetrou coordinates  $x^\mu = (t, \rho, z, \phi)$ , the simplest form of the line element for the stationary axisymmetric case was given by Papapetrou [55],

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (1)$$

with  $g_{tt} = -f(\rho, z)$ ,  $g_{t\phi} = f(\rho, z)\omega(\rho, z)$ ,  $g_{\phi\phi} = \rho^2 f^{-1}(\rho, z) - f(\rho, z)\omega^2(\rho, z)$  and  $g_{zz} = g_{\rho\rho} = e^{2\gamma(\rho, z)} f^{-1}(\rho, z)$ . The metric functions  $f$ ,  $\omega$  and  $\gamma$  can be obtained from the Ernst complex potentials  $\mathcal{E}$  and  $\Phi$  (see details in Ref. [46]). The Ernst potentials obey the relations [46]

$$\begin{aligned} (\text{Re } \mathcal{E} + |\Phi|^2) \nabla^2 \mathcal{E} &= (\nabla \mathcal{E} + 2\Phi^* \nabla \Phi) \cdot \nabla \mathcal{E}, \\ (\text{Re } \mathcal{E} + |\Phi|^2) \nabla^2 \Phi &= (\nabla \mathcal{E} + 2\Phi^* \nabla \Phi) \cdot \nabla \Phi. \end{aligned} \quad (2)$$

From a physical viewpoint, the Ernst potentials are relevant because they lead to the definition of the analogues of the Newtonian gravitational potential,  $\xi = (1 - \mathcal{E})/(1 + \mathcal{E})$ , and the Coulomb potential,  $q = 2\Phi/(1 + \mathcal{E})$ . The real part of  $\xi$  accounts for the matter distribution and its imaginary part for the mass currents. Besides, the real part of the  $q$  potential denotes the electric field and its imaginary part the magnetic field.

The Ernst equations (2) can be solved by means of the Sibgatullin's integral method [47, 48], according to which the complex potentials  $\mathcal{E}$  and  $\Phi$  can be calculated from specified axis data  $\mathcal{E}(z, \rho = 0)$  and  $\Phi(z, \rho = 0)$  [47, 48]. Motivated by the accuracy [41, 42] and the level of generality of the analytic solution derived in Ref. [42], the Ernst potential  $\mathcal{E}(z, \rho = 0)$  is chosen as in Ref. [42]. To construct the exact solution that represents the electromagnetic field configuration described above, the Ernst potential  $\Phi(z, \rho = 0)$  is chosen following the prescription in Ref. [49]. The Ernst potentials on the symmetry axis read

$$\begin{aligned} \mathcal{E}(z, \rho = 0) &= \frac{z^3 - z^2(m + ia) - kz + is}{z^3 + z^2(m - ia) - kz + is}, \\ \Phi(z, \rho = 0) &= \frac{z^2\zeta + z(v + i\mu) + i\zeta + \chi}{z^3 + z^2(m - ia) - kz + is}. \end{aligned} \quad (3)$$

The physical meaning of the parameters in Eq. (3) is

derived from the multipole moments calculated using

the Fodor-Hoenselaers-Perjés procedure [56] (see also Ref. [57]). For the present case,

$$\begin{aligned}
P_0 &= m, & P_1 &= iam, & P_2 &= (k - a^2)m, & P_3 &= -im(a^3 - 2ak + s), \\
P_4 &= \frac{1}{70}m [70a^4 - 210a^2k + 13a\varsigma(\mu - iv) + 140as + 10k(7k - m^2 + \varsigma^2) + 3(\mu^2 + v^2)], \\
P_5 &= -\frac{1}{21}im \{ -21a^5 + 84a^3k + a^2(-6\mu\varsigma - 63s + 6iv\varsigma) \\
&\quad + a[-63k^2 + 6k(m - \varsigma)(m + \varsigma) + \mu^2 + 5i\varsigma\varsigma + v^2 + 5\chi\varsigma] \\
&\quad + k(3\mu\varsigma + 42s - iv\varsigma) + 2i(\mu\varsigma + v\chi) + 7s(\varsigma^2 - m^2) \},
\end{aligned} \tag{4}$$

$$\begin{aligned}
Q_0 &= \varsigma, & Q_1 &= v + i(a\varsigma + \mu), & Q_2 &= -a^2\varsigma - a\mu + k\varsigma + \chi + i(av + \varsigma), \\
Q_3 &= -a^2v - a\varsigma + kv + i(-a^3\varsigma - a^2\mu + a(2k\varsigma + \chi) + k\mu - s\varsigma), \\
Q_4 &= a^4\varsigma + a^3(\mu - iv) + a^2(-3k\varsigma - i\varsigma - \chi) + \frac{1}{70}a [140s\varsigma - (\mu - iv)(140k - 3m^2 - 10\varsigma^2)] \\
&\quad + \frac{1}{7} \{ \varsigma[\varsigma(k\varsigma + i\varsigma + \chi) + (\mu - iv)^2] + (7k - m^2)(k\varsigma + i\varsigma + \chi) + 7s(\mu - iv) \}, \\
Q_5 &= \frac{1}{21} \{ 21ia^5\varsigma + 21a^4(v + i\mu) - 21ia^3(4k\varsigma + i\varsigma + \chi) + a^2[63is\varsigma - i(63k + 8\varsigma^2)(\mu - iv)] \\
&\quad + ia \{ -\varsigma[8k\varsigma + i\varsigma + \chi] + 9\mu^2 - 16i\mu v - 7v^2 \} + (21k - 2m^2)(3k\varsigma + 2i\varsigma + 2\chi) \\
&\quad + 42s(\mu - iv) \} - i(\mu - iv)(-21k^2 + 2km^2 + \mu^2 + v^2) + k\varsigma^2(v - i\mu) \\
&\quad + \varsigma(-42iks - 6\mu\varsigma + 6i\mu\chi + 7im^2s + 8i\varsigma v + 8v\chi) + 21s(\varsigma - i\chi) + 7is\varsigma^3 \}.
\end{aligned} \tag{5}$$

Symbol	Associated Multipole Moment
$\varsigma$	Electric monopole
$v$	Electric dipole
$\chi$	Electric quadrupole
$\mu$	Magnetic dipole
$\zeta$	Magnetic quadrupole

Table I. Summary of the electromagnetic parameters and the multipole moments they are related to.

Specifically, the interpretation of the parameters based on the multipole expansion in Eqs. (4) and (5) is as follows. The real parameter  $m$  corresponds to the total mass,  $a$  to the total angular momentum per unit mass while  $k$  and  $s$  are related to the mass-quadrupole moment and the differential rotations, respectively. For later convenience, an electric monopole contribution  $Q_0$ , characterized by the parameter  $\varsigma$ , was introduced above. Parameters  $v$  and  $\mu$  are associated with the electric and magnetic dipole moments, respectively, whereas  $\chi$  and  $\zeta$  with the electric and magnetic quadrupole moments, respectively. The existence of an electric dipole is theorized for strange stars (See Ref. [58]). A summary of the arbitrary parameters and the multipole moments they are related to can be found in Table I.

The mass moment  $P_2$  governs the deformation of the star and it is composed of two parts: the term  $a^2m$  that

is the usual rotation-induced deformation and a second contribution  $km$  that accounts for a possible intrinsic deformation of the star [59, 60]. An analogous argument can be formulated in the case of the electric moments. The real part of  $Q_2$  accounts for the electric quadrupole contribution to the total electromagnetic quadrupole moment. The terms  $-a^2\varsigma$  and  $-a\mu$  account for the rotation-induced redistribution of the electric charge and deformation of the magnetic dipole; whereas, the term  $k\varsigma$  accounts for the contribution from the charge distributed over the intrinsic deformed mass. The additional parameter  $\chi$  is added to account for any additional possible contribution to the total quadrupole moment.

The multipole expansion in Eq. (5) shows that even if the magnetic dipole parameter is zero ( $\mu = 0$ ), a magnetic dipole component (imaginary part of  $Q_1$ ) is present provided by the rotation of the electric charge  $Q_0$ . Similarly, even if the magnetic quadrupole parameter  $\zeta$  is set to zero, a rotating electric dipole can induce a magnetic quadrupole (imaginary part of  $Q_2$ ). For the electric part of the multipole expansion in Eq. (5), an analogous behavior is observed, namely, a rotating magnetic dipole can induce an electric quadrupole moment and a rotating magnetic quadrupole can generate an electric octupole moment. Based on these processes, the astrophysical source can afford a non-vanishing induced Poynting vector and, correspondingly, an induced non-vanishing flux of electromagnetic energy around the source that will

contribute to the frame dragging induced solely by the mass currents.

#### IV. CHARACTERIZATION OF THE ELECTROMAGNETIC FIELDS

To describe the electromagnetic properties of the solution, the electric and magnetic fields, in the spacetime surrounding the star, are calculated by means of the expressions

$$E_\alpha = F_{\alpha\beta}u^\beta, \quad B_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta}F_{\gamma\delta}u^\beta, \quad (6)$$

where  $F_{\alpha\beta}$  is the electromagnetic field tensor  $F_{\alpha\beta} = 2A_{[\beta;\alpha]}$ ,  $A_\mu = (0, 0, A_\phi, -A_t)$  is the electromagnetic four-potential,  $u_\alpha$  is a time-like vector and  $\epsilon_{\alpha\beta\gamma\delta}$  is the totally antisymmetric tensor of positive orientation with norm  $\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\alpha\beta\gamma\delta} = -24$  [61]. For a congruence of observers at rest in the frame of (1), the four-velocity is defined by the time-like vector  $u^\alpha = (1/\sqrt{f}, 0, 0, 0)$ . The vectorial fields have components in the  $\rho$  and  $z$  directions only. The components of the electric field are given by

$$E_\rho = -\frac{\sqrt{f}}{e^{2\gamma}}A_{t,\rho}, \quad E_z = -\frac{\sqrt{f}}{e^{2\gamma}}A_{t,z}, \quad (7)$$

and for the magnetic field by

$$B_\rho = \frac{f^{3/2}}{\rho e^{2\gamma}}(-\omega A_{t,z} + A_{\phi,z}), \quad (8)$$

$$B_z = -\frac{f^{3/2}}{\rho e^{2\gamma}}(-\omega A_{t,\rho} + A_{\phi,\rho}). \quad (9)$$

The explicit form of the fields can be found in Appendix B.

Figure 1 shows the force lines of the magnetic field for various values of the magnetic quadrupole parameter  $\zeta$  and for realistic values of the mass and mass current multipoles. Specifically, the vacuum multipole moments of the solution mass, angular momentum and mass quadrupole and current octupole have been fixed to the numerical ones obtained in Ref. [62]. They are listed in Table II. In particular, for Fig. 1.a,  $\zeta = 0 \text{ km}^3$ ; 1.b,  $\zeta = 10 \text{ km}^3$ ; 1.c,  $\zeta = 25 \text{ km}^3$  and for Fig. 1.d,  $\zeta = 50 \text{ km}^3$ . The increasing of the separation between consecutive force lines indicates that the magnetic field decreases while the distance increases. Figure 1 not only shows how the reflection symmetry around the plane  $z = 0$  is broken because of the magnetic quadrupole [6], but also shows that at large distances from the source, the magnetic field behaves like a magnetic dipole despite the presence of strong non-dipolar contributions.

Figure 2 shows the force lines of the electric field for a variety of values of the electric quadrupole for realistic values of the mass and mass current multipoles listed

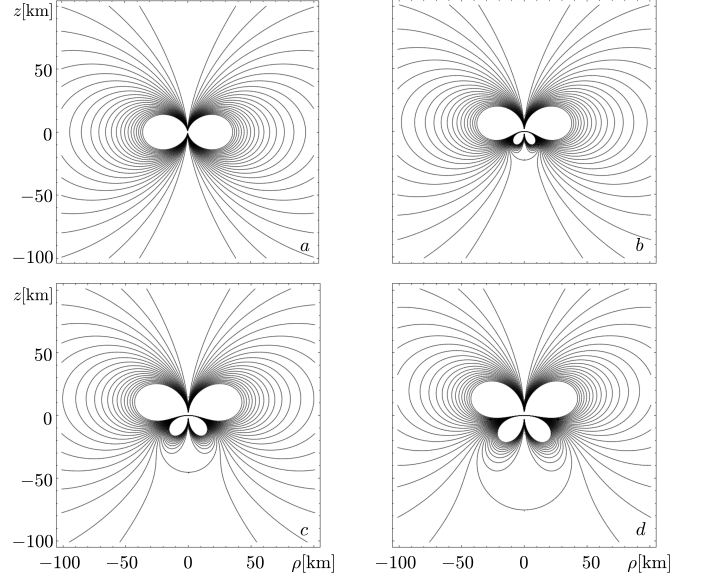


Figure 1. Magnetic field force lines for  $m = 2.071 \text{ km}$ ,  $j = 0.194$ ,  $Q = -2.76 \text{ km}^3$ ,  $s = -2.28 \text{ km}^4$ ,  $\varsigma = 0 \text{ km}$ ,  $v = 0 \text{ km}^2$ ,  $\chi = 0$ ,  $\mu = 1 \text{ km}^2$  for (a)  $\zeta = 0 \text{ km}^3$ , (b)  $\zeta = 15 \text{ km}^3$ , (c)  $\zeta = 30 \text{ km}^3$  and (d)  $\zeta = 50 \text{ km}^3$ . The non-electromagnetic parameters correspond to the model 2 for the equation of state L in Ref. [62] (see also Table II).

Model	$m$ [km]	$j$	$Q$ [km <sup>3</sup> ]	$s$ [km <sup>4</sup> ]
M2	2.071	0.194	-2.76	-2.28
M3	2.075	0.324	-7.55	-10.5
M4	2.080	0.417	-12.2	-22.0
M5	2.083	0.483	-16.2	-33.9

Table II. Realistic numerical solutions for rotating neutron stars derived by Pappas and Apostolatos [62] for the equation of state L. Here,  $m$  is the total mass of the star,  $j$  is the dimensionless spin parameter:  $j = J/M^2$  (being  $J$  the angular momentum),  $Q$  is the quadrupole moment and  $s$  is the current octupole moment. See Table VI of [62].

in Table II. Specifically, for Fig. 2.a,  $\chi = 0 \text{ km}^3$ ; 2.b,  $\chi = 10 \text{ km}^3$ ; 2.c,  $\chi = 25 \text{ km}^3$  and for Fig. 2.d,  $\chi = 50 \text{ km}^3$ . As above, it is clear that reflection symmetry is broken and that at large distances the electric dipole component dominates the field configuration.

In general, the electromagnetic field of astrophysical objects is expected to be a combination of the results depicted in Figs. 1 and 2. Moreover, as shown below, the breaking of the reflection symmetry has an important role on the vorticity of the spacetime. After characterize the electromagnetic fields, the generation of purely electromagnetic frame dragging is discussed below.



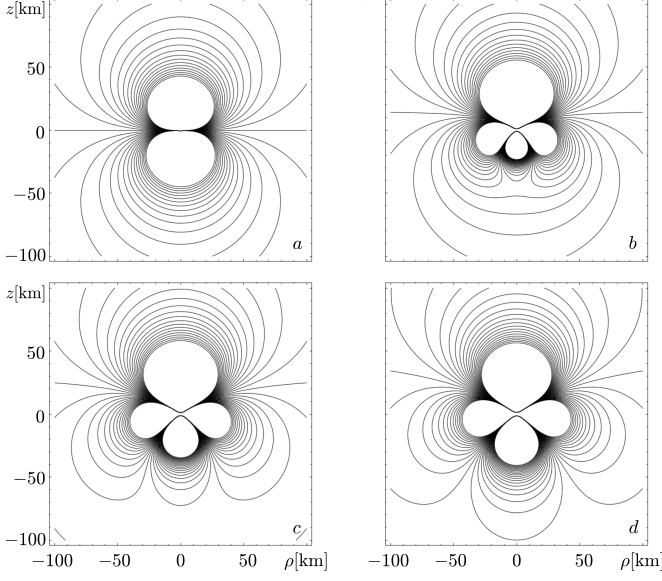


Figure 2. Electric field force lines for  $m = 2.071$  km,  $j = 0.194$ ,  $Q = -2.76$  km<sup>3</sup>,  $s = -2.28$  km<sup>4</sup>,  $\zeta = 0$  km,  $v = 1$  km<sup>2</sup>,  $\mu = 0$  km<sup>2</sup>,  $\zeta = 0$  for (a)  $\chi = 0$  km<sup>3</sup>, (b)  $\chi = 10$  km<sup>3</sup>, (c)  $\chi = 25$  km<sup>3</sup> and (d)  $\chi = 50$  km<sup>3</sup>.

## V. VORTICITY SCALAR AND POYNTING VECTOR

The physics of the frame dragging states that the rotation of the source induces a twist in the neighbourhood that drags any frame of reference near the source. In the case of spacetimes endowed by complex electromagnetic fields and mass currents, frame dragging originates from a combination of the vorticity of the electromagnetic field and the vorticity associated with the mass currents.

*Poynting vector.*—The Poynting vector  $\mathbf{S}$  carries the information about the electromagnetic energy flux in the spacetime. Because of the axially symmetric character of the spacetime, only the component along the unitary vector  $\hat{e}_\phi$  survives [6]. In terms of the Ernst potentials  $S^\phi = \sqrt{f} \text{Im}(\Phi^* \Phi_{,z}) / (4\pi \rho e^{2\gamma})$  [63] or more conveniently

$$S^\phi = \frac{\sqrt{f}}{8\pi \rho e^{2\gamma}} \nabla \Phi^* \times \nabla \Phi, \quad (10)$$

where it is clear that  $S^\phi$  vanishes if  $\Phi$  is purely real, purely imaginary or when their real and imaginary parts are proportional to each other. Due to the complex combination of the electromagnetic moments with the mass currents, their effects on the Poynting vector (and subsequently to the vorticity) are often subtle. However, the multipole expansion in Eq. (5) allows for a detailed description of each contribution. For instance, if  $v = 0$ ,  $\chi = 0$  and  $|a\zeta| < |\mu|$ , it is then clear that, to leading order, a sign change in the magnetic dipole moment parameter  $\mu$  changes the sign of the magnetic field. This has the effect of changing the rotation direction of the Poynting vector (see Fig. 3) and

as discussed below, it could decrease the total vorticity of the spacetime. Figure (3) depicts the Poynting vector circulation around the source, when  $\mu$  is chosen parallel (l.h.s. panel) and antiparallel (r.h.s. panel) to the star's rotation axis.

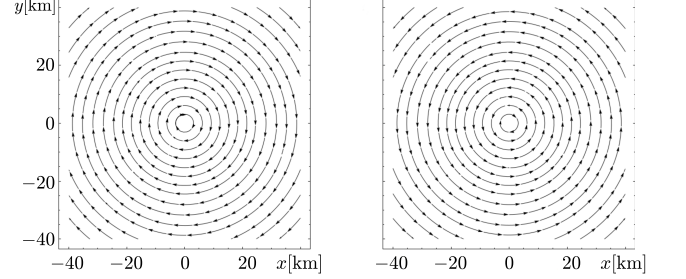


Figure 3. Vector field of the Poynting vector for the star model  $m = 2.071$  km,  $j = 0.194$ ,  $Q = -2.76$  km<sup>3</sup>,  $s = -2.28$  km<sup>4</sup>. The electromagnetic multipoles were chosen zero except  $\zeta = 0.1$  km,  $\mu = 1$  km<sup>2</sup> in the left panel and  $\mu = -1$  km<sup>2</sup> in the right side. The direction of the Poynting vector is due the sign change in the multipole expansions in Eq. (5). Figures are presented in the quasi-Cartesian auxiliary coordinates  $r \rightarrow \sqrt{x^2 + y^2}$  and  $\phi = \arctan(y/x)$ .

The change of the circulation of the Poynting vector can be understood from the potential  $q(\rho, z)$  that is analogous to the Coloumb potential (see above). Under the condition that the reflection symmetry is not broken,  $v = 0$  and  $\zeta = 0$ , a sign change of the magnetic dipole parameter  $\mu$  is equivalent to change the sign of the coordiante  $z$  with a global minus sign, namely,  $q(\rho, z; v = 0, \zeta = 0, -\mu) = -q(\rho, -z; v = 0, \zeta = 0, \mu) = -q^*(\rho, z; v = 0, \zeta = 0, \mu)$ . Therefore, under the conditions that reflection symmetry imposes [49], all the electric field moments change sing and this leads to a global sing of the Poynting vector.

*Vorticity Scalar.*—For an observer at rest respect to (1), the vorticity tensor is defined by  $\omega_{\alpha\beta} = u_{[\alpha;\beta]} + \dot{u}_{[\alpha} u_{\beta]}$ . The direct calculation of  $\omega_{\alpha\beta}$  for (1) yields to [6]

$$\omega_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{f}\omega_{,\rho} \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{f}\omega_{,z} \\ 0 & \frac{1}{2}\sqrt{f}\omega_{,\rho} & \frac{1}{2}\sqrt{f}\omega_{,z} & 0 \end{pmatrix}. \quad (11)$$

To quantify the vorticity, it is convenient to introduce the vorticity scalar that is defined by the contraction of the vorticity tensor and reads

$$\omega_v = (\omega_\beta^\alpha \omega_\alpha^\beta)^{\frac{1}{2}} = f \sqrt{f(\omega_{,\rho}^2 + \omega_{,z}^2)} / \sqrt{2e^{2\gamma}\rho^2}, \quad (12)$$

with  $\omega_{,\rho} = -\rho f^{-2} \Im(\mathcal{E}_{,z} + 2\Phi^* \Phi_{,z})$  and  $\omega_{,z} = \rho f^{-2} \Im(\mathcal{E}_{,\rho} + 2\Phi^* \Phi_{,\rho})$ . More explicitly, it reads

$$\omega_v = \frac{e^{-\gamma}}{\sqrt{2}f} \sqrt{\Im[\mathcal{E}_{,z} + 2\Phi^* \Phi_{,z}]^2 + \Im[\mathcal{E}_{,\rho} + 2\Phi^* \Phi_{,\rho}]^2} \quad (13)$$

The vorticity scalar can be understood in terms of its fluid mechanics analogue. It represents the rotation of the fluid. In the general relativistic case, it can be related to the rotation velocity of a family of congruences.

Equation (13) is very useful to analyze the contributions to the vorticity because the electromagnetic and mass currents terms can be easily identified there. Moreover, this identification can be accompanied by further expressing these contributions in terms of the parameters of the Ernst potentials (3). In particular, in absence of electromagnetic fields, the imaginary part of  $\mathcal{E}$  is associated with mass currents. Thus, for static sources one could attempt to assume  $\mathcal{E}$  as real in Eq. (13) and focus only in the electromagnetic contributions encoded in  $\Phi$ . However, because  $\mathcal{E}$  and  $\Phi$  are not independent objects [see, e.g., Eq. (2)], in the presence of electromagnetic fields,  $\mathcal{E}$  and  $\Phi$  are complex even for non-spinning astrophysical objects. Notwithstanding, if  $\Phi$  is purely real or purely imaginary, for non-rotating objects, the imaginary part of  $\mathcal{E}$  vanishes [see, e.g., Eq. (2)]. Therefore, interest here is in identifying when the electromagnetic contribution does not vanish.

In doing so, it is convenient to express  $\omega_v = |\Im(\nabla\mathcal{E} + 2\Phi^*\nabla\Phi)|/(\sqrt{2}fe^\gamma)$ , so that the term  $\Im\Phi^*\nabla\Phi$  vanishes if  $\Phi$  is purely real, purely imaginary or when their real and imaginary parts are proportional to each other. Based on the discussion above, see also Eq. (5), the purely real case corresponds to the absence of magnetic fields and the purely imaginary case to the absence of electric fields. In the cases when the Poynting vector is zero, the electromagnetic contribution to the vorticity vanishes; this generalizes the results in Ref. [6] where a particular spacetime was considered. The particular case of proportional real and imaginary parts leads to the case of proportional electric and magnetic fields and corresponds to the cases studied in Refs. [4, 6].

Note that the considerations above on the vorticity scalar and the Poynting vector are completely general and valid for any stationary axially symmetric spacetime.

To study the contribution of the electromagnetic energy flux to the vorticity, characterize first the contributions of the rotation of the source. In this case, the Ernst potential  $\Phi$  that encodes all the electromagnetic moments is zero and  $\mathcal{E}$  is complex. For fast rotating stars, the vorticity scalar (and the Lense-Thirring effect) is larger than for slow rotating stars, this can be seen in the Fig. (4).

When one includes an electromagnetic field to a particular star model, the Ernst potential  $\Phi$  is no longer zero, and it has an important role in the vorticity scalar. The particular contribution depends on the structure generated by the multipole expansion and the Poynting vector. Before considering the purely electromagnetic contribution to the frame-dragging in realistic situations (zero or negligible total electric charge), consider the case of a source with a dipole magnetic field and electric charge.

The upper panel of Fig. (5) depicts the functional dependence of the vorticity scalar on the distance from the

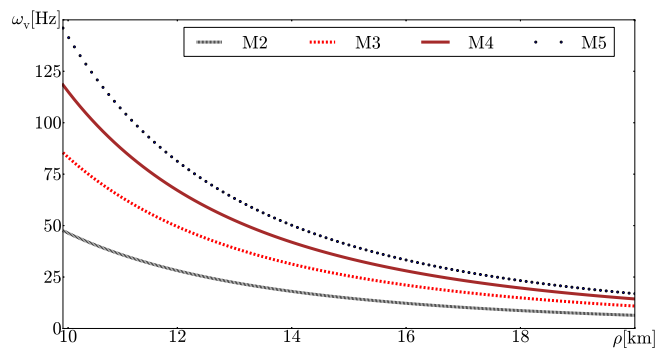


Figure 4. Vorticity generated by the mass currents. All the electromagnetic moments are zero in this case. The vorticity of the spacetime was calculated for the models 2 ( $j = 0.194$ ), 3 ( $j = 0.324$ ), 4 ( $j = 0.417$ ) and 5 ( $j = 0.483$ ) with the equation of state L in Ref. [62]. They are listed in the Table II

star for a fixed value of the electric monopole and for a variety of magnetic dipole moments. The vorticity scalar decreases (increases) from its value in the vacuum situation ( $\mu = 0$ ) when the dipole is parallel (anti-parallel) to the star's rotation-axis. The reason for this phenomenon can be understood in terms of Eq. (13). To do so, note that  $\mathcal{E}$  decreases monotonically as a function of  $z$  and  $\rho$ ; thus, its derivatives carries a negative sign. Although the same argument applies to  $\Phi$ , when the sign of  $\mu$  changes, see above, the Poynting vector changes its circulation and that is enough to change the sign of its contribution to the vorticity scalar [see Eq. (13)]. Alternatively, assume that the reflection symmetry exists and set the mass currents parameters  $a$  and  $s$  to zero, for this case,  $\Phi(\rho, z; a = 0, s = 0, -\mu) = \Phi^*(\rho, z; a = 0, s = 0, \mu)$  and therefore  $\Im\Phi^*\nabla\Phi$  changes sign.

The lower panel of Fig. 5 depicts the functional dependence of the vorticity scalar on the distance from the star for a fixed value of the electric dipole and for a variety of magnetic dipole moments. The contribution in this case is weaker than in the case of an electric monopole. However, results in the lower panel of Fig. 5 are more realistic than those in the upper panel. Interestingly, in the lower panel, there is no change in the sign of the electromagnetic contribution to the vorticity scalar when  $\mu$  changes sign. The reason for this relies on the fact that for  $v \neq 0$  or  $\zeta \neq 0$ , the reflection symmetry around the equatorial plane breaks down and the arguments provided above do not apply, i.e.,  $\Phi(\rho, z; a = 0, s = 0, -\mu)$  does not equate to  $\Phi^*(\rho, z; a = 0, s = 0, \mu)$ .

Based on the processes described above on the generation of a Poynting vector from the fields induced by the mass currents ( $a \neq 0$  and  $s \neq 0$ ), it is clear that even if the parameters associated with the electric field ( $\zeta$ ,  $v$  and  $\chi$ ) are set to zero, but the parameters associated with the magnetic field ( $\mu$  and  $\zeta$ ) are non-vanishing, then a rotationally-induced electromagnetic contribution to the vorticity scalar is present. A similar scenario takes

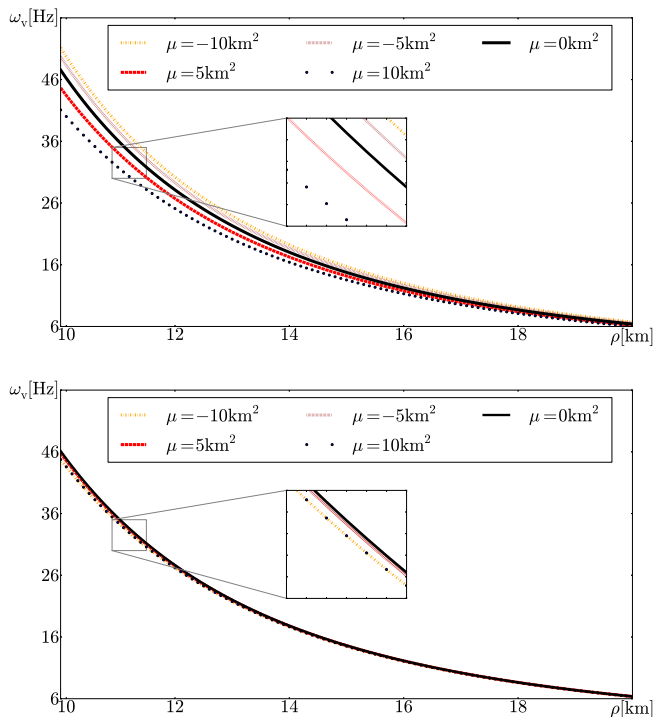


Figure 5. Electromagnetically generated vorticity by the presence of a magnetic dipole and a fixed electric monopole of  $\zeta = 0.1$  km (upper panel with  $v = 0$ ,  $\chi = 0$  and  $\zeta = 0$ ) and an electric dipole of  $\mu = 10$  km<sup>2</sup> (lower panel with  $\zeta = 0$ ,  $\chi = 0$  and  $\zeta = 0$ ). The vorticity of the spacetime was calculated for the model 2 for the equation of state L in Ref. [62]. Parameters are listed in Table II.

place if the parameters associated with the magnetic field are set to zero and non-vanishing electric parameters are considered. In particular, even in the low rotation regime, a non-negligible electric field is generated [64] and may be important to characterize the evolution of the electromagnetic structure in neutron stars. Moreover, in the case of fast rotation, the frame-dragging caused by a Kerr black hole significantly distorts the structure of an external magnetic field and this scenario may be relevant for low accreting black holes as the one present in the Milky Way center [65, 66].

The most relevant contributions to the vorticity scalar from the electromagnetic field were considered above. However, in general, all the multipole moments contribute to the vorticity scalar and the details of the net result may deviate from those discussed above; albeit, the magnitude of the effect is not expected to differ from the predictions in Fig. 5.

## VI. ORBITAL EQUATORIAL MOTION AND EPICYCLIC FREQUENCIES

In general relativity, as in Newtonian gravitation, all the physical characteristics of the source have an effect

on the dynamics of orbiting of particles. One of the most distinctive points between both theories is the frame-dragging, but is not an observable by itself. A way to measure it, is to appeal to the dynamics around the source and characterizing the behavior of, e.g., neutral test particles [40, 53].

Observationally, the Keplerian motion of matter could be useful to model the quasi-periodic oscillations (QPOs) that are present in the low-mass X-ray binaries (LMXBs) containing a neutron star [67]. These oscillations occur at frequencies in the range of kHz and come in pairs, the upper and the lower mode correspond to the frequencies of Keplerian motion and periastron precession of the accreted matter in the close vicinity of the star [52]. An additional effect is that these equatorial orbits will exhibit a relativistic nodal precession due to frame dragging [53], causing a detectable signal in the spectra of LMXBs [68].

The following subsections are devoted to quantifying the magnetic quadrupole effect in the ISCO and the epicyclic frequencies for different models of neutron stars [62].

### Influence of the field on the ISCO radii

The dynamics of an orbiting particle can be analyzed by using the Lagrangian formalism. To do so, consider a particle of rest mass  $m_0 = 1$  moving in the space-time described by the metric functions in Eq. (1), the Lagrangian of the particle is then given by

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \quad (14)$$

where the dot denotes differentiation with respect to the proper time  $\tau$ ,  $x^\mu(\tau)$  are the Weyl-Lewis-Papapetrou coordinates. For a stationary and axisymmetric spacetime, there are two constants of motion related to the time coordinate  $t$  and azimuthal coordinate  $\phi$ . Thus, the energy and the canonical angular momentum, respectively, are conserved (see Ref. [69] for details). Assuming that the motion takes place in the equatorial plane of the star  $z = 0$ , and taking into account the four velocity normalization for massive particles  $g_{\mu\nu} u^\mu u^\nu = -1$ , one can identify an effective potential that characterizes the motion in the plane (see, e.g., Ref. [70])

$$V_{\text{eff}}(\rho) = 1 - \frac{E^2 g_{\phi\phi} + 2ELg_{t\phi} + L^2 g_{tt}}{g_{t\phi}^2 - g_{\phi\phi} g_{tt}}. \quad (15)$$

For a particle moving in a circular orbit, the energy  $E$  and the canonical angular momentum  $L$  are determined by the conditions  $V_{\text{eff}}(\rho) = 0$  and  $dV_{\text{eff}}/d\rho = 0$  (see, e.g., Ref. [71]). The marginal stability condition reads  $d^2V_{\text{eff}}/d\rho^2 = 0$ . Thus, the ISCO's radius is determined by solving numerically the previous equation for  $\rho$ .

The importance of determining the ISCO is that accretion disks extend from the last stable orbit to exterior zones, so the ISCO is an inner boundary for the accreted matter. As stated above, the inclusion of a magnetic

field causes a dependence of all physical quantities of the intensity on the field. From the International System of Units (SI), the conversion to geometrized units is given by

$$\mu_{\text{geom}} = \frac{10^{-6} \sqrt{G\mu_0}}{c^2} \mu_{\text{SI}}, \quad (16)$$

where  $G$  is the gravitational constant,  $\mu_0$  is the vacuum permittivity,  $c$  is the speed of light and  $\mu$  is given in units of  $\text{Am}^2$ . The units of  $\zeta$  are  $\text{Am}^3$  and the conversion for the quadrupole reads

$$\zeta_{\text{geom}} = \frac{10^{-9} \sqrt{G\mu_0}}{c^2} \zeta_{\text{SI}}. \quad (17)$$

In the same way, the electric multipole moments can be written in geometrized units as:

$$\varsigma_{\text{geom}} = \sqrt{\frac{G\mu_0}{4\pi}} \varsigma_{\text{SI}}, \quad (18)$$

$$v_{\text{geom}} = 10^{-3} \sqrt{\frac{G\mu_0}{4\pi}} v_{\text{SI}}, \quad (19)$$

$$\chi_{\text{geom}} = 10^{-6} \sqrt{\frac{G\mu_0}{4\pi}} \chi_{\text{SI}}. \quad (20)$$

Consider a superposition of a fixed dipolar and quadrupole component that will be varied from  $1 \text{ km}^3$  to  $50 \text{ km}^3$ . Figure 6 depicts the ISCO radius as a function of the parameter  $\mu$  and  $\zeta$  for three realistic numerical solutions for rotating neutron stars models derived in Ref. [62]. The parameter  $\mu$  is set to  $1 \text{ km}^2$  that corresponds to a magnetic dipole field of  $10^{12} \text{ T}$ . The quadrupole parameter  $\zeta$  is chosen between 0 and  $50 \text{ km}^3$  that corresponds to magnetic quadrupoles from 0 to  $5 \times 10^{35} \text{ Am}^3$ , respectively. In all these cases, the ISCO radius decreases for

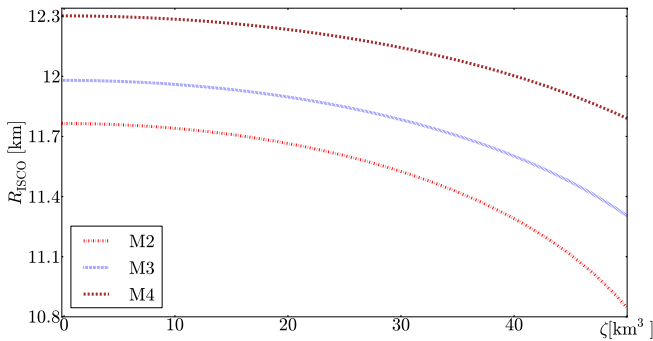


Figure 6. ISCO radius as a function of magnetic quadrupole parameter  $\zeta$ , with  $\mu = 0 \text{ km}^2$ . The physical parameters for the star correspond to the models 2-4 listed in Table II. We can see that the ISCO radius decreases for increasing  $\zeta$ .

increasing  $\zeta$ . This can be explained as a result of the deformation of the space-time by the energy stored in the electromagnetic (see, e.g., Ref. [40]). We elaborate more

on this in the next section.

### Influence of the electromagnetic field in the Keplerian and the epicyclic frequencies

Imposing the conditions of constant orbital radius,  $d\rho/d\tau = 0$  and taking into account that  $d\phi/d\tau = \Omega_K dt/d\tau$ , the Keplerian frequency reads (see, e.g., Ref. [69])

$$\Omega_K = \frac{d\phi}{dt} = \frac{-g_{t\phi,\rho} \pm \sqrt{(g_{t\phi,\rho})^2 - g_{\phi\phi,\rho} g_{tt,\rho}}}{g_{\phi\phi,\rho}}, \quad (21)$$

where “+” and “−” denotes the Keplerian frequency for corotating and counter-rotating orbits, respectively. In Ref. [40] the functional dependence of the Keplerian frequency on the magnetic dipole parameter  $\mu$  was discussed, here interest is in the functional dependence on the magnetic quadrupole parameter  $\zeta$ . The upper panel of Fig. 7 depicts the functional dependence on  $\zeta$ , as in the case analyzed in Ref. [40], the Keplerian frequency increases with increasing  $\zeta$  because the ISCO’s radius decreases (see Fig. 6) as a consequence of the additional deformation of the space-time by the electromagnetic field. In particular, the energy of the electromagnetic field shifts the marginally unstable region toward the star.

Analytical expressions for the radial and vertical frequencies follow from allowing slightly radial and vertical perturbations of the orbit. According to Ref. [69], the radial and vertical epicyclic frequencies are given by

$$\nu_\alpha = \frac{1}{2\pi} \left\{ -\frac{g^{\alpha\alpha}}{2} \left[ (g_{tt} + g_{t\phi}\Omega_k)^2 \left( \frac{g_{\phi\phi}}{\rho^2} \right)_{,\alpha\alpha} - 2(g_{tt} + g_{t\phi}\Omega_k)(g_{t\phi} + g_{\phi\phi}\Omega_k) \left( \frac{g_{t\phi}}{\rho^2} \right)_{,\alpha} + (g_{t\phi} + g_{\phi\phi}\Omega_k)^2 \left( \frac{g_{tt}}{\rho^2} \right)_{,\alpha\alpha} \right] \right\}, \quad (22)$$

with  $\alpha = \{\rho, z\}$ . In the relativistic precession model (RPM) [51], the periastron  $\nu_\rho^p$  and the nodal  $\nu_z^p$  frequencies are the observational relevant ones, they are defined by

$$\nu_\alpha^p = \frac{\Omega_K}{2\pi} - \nu_\alpha. \quad (23)$$

At the ISCO, the radial oscillation frequency equals the Keplerian frequency and only the vertical precession  $\alpha = z$  is considered here. The influence of the magnetic quadrupole on the vertical precession frequency is depicted in the lower panel of Fig. 7. The behaviour and the underlying physical mechanism are analogous to those for the Keplerian frequency.

The frequency  $\nu_{\text{LT}}$  that characterizes the Lense–Thirring effect is given by (see, e.g., Ref. [69])

$$\nu_{\text{LT}} = -\frac{1}{2\pi} \frac{g_{t\phi}}{g_{\phi\phi}}, \quad (24)$$



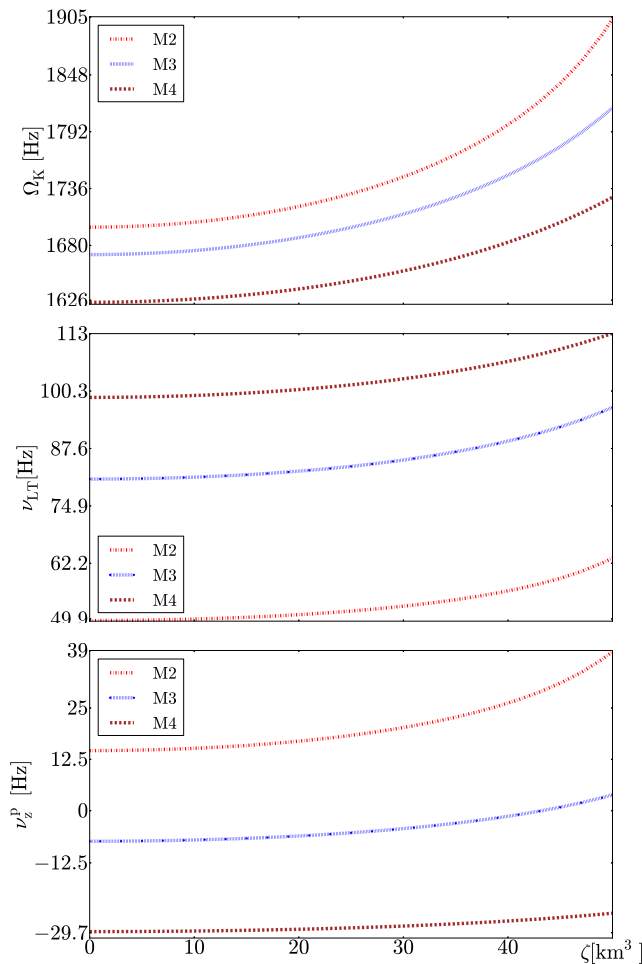


Figure 7. Influence of the magnetic quadrupole moment  $\zeta$  in the epicyclic frequencies for different models of neutron stars with  $\varsigma = 0$ ,  $\mu = 1 \text{ km}^2$ ,  $v = 0$  and  $\chi = 0$ . Vacuum parameters of the star are listed in Table II.

and could be relevant to model the horizontal branch oscillations observed in the Low Mass X-Ray Binaries (LMXBs) [51]. The strength of the frame-dragging effect increases for fast rotating objects and, as shown in the central panel of Fig. 7, for stronger magnetic fields. Figure 7 also shows that the electromagnetically-induced frame dragging has a direct effect on the orbiting particles, even if they are neutral.

## VII. CONCLUDING REMARKS

We present a new stationary axisymmetric nine-parameter closed-form analytic solution that generalizes the Kerr solution with arbitrary mass-quadrupole moment, octupole current moment, electric and magnetic dipole and electric and magnetic quadrupole moments. The analytic form of its multipolar structure and their electric and magnetic fields are also presented. According to the arguments presented through the paper, this solution could be used to model the exterior gravitational and electromagnetic fields around strongly magnetized stars, in particular white dwarfs. Besides, this model also could be used even for the description of exotic stars such as the  $\tau$  Sco recently reported by Donati *et al.* [72].

This solution allowed for a comprehensive analysis of the contribution of complex and intense electromagnetic fields to quintessential hallmark of general relativity, namely, the Lense–Thirring effect. It was shown that if the value of the parameters is such that the reflection symmetry is preserved [49], then a sign change of the magnetic dipole parameter  $\mu$  is enough to change the direction of the circulation of the Poynting vector and the sign of the electromagnetic contribution to the vorticity scalar. The influence of complex electromagnetic fields on the ISCO's radius and the epicyclic frequencies were also considered and it was shown that for strong magnetic fields, the influence is not negligible.

Additional interest is in studying the effect of the magnetic field in the quadrupole moment of the star (see, e.g., Refs. [73, 74]). This topic has gained recently theoretical and observational interest and is currently discussed under the title of  $I - \text{Love} - Q$  relations [74]. Due to the great generality of the present analytic exact solution, this subject will be explored in a forthcoming contribution [75].

## ACKNOWLEDGMENTS

Fruitful discussions with Prof. César A. Valenzuela-Toledo are acknowledged with pleasure. This work was partially supported by Fundación para la Promoción de la Investigación y la Tecnología del Banco la República Grant No. 2879. and by Comité para el Desarrollo de la Investigación (CODI) of Universidad de Antioquia under the Estrategia de Sostenibilidad 2015-2016.

[1] C. Everitt, D. DeBra, B. Parkinson, J. Turneare, J. Conklin, M. Heifetz, G. Keiser, A. Silbergleit, T. Holmes, J. Kolodziejczak, M. Al-Meshari, J. Mester, B. Muhlfelder, V. Solomonik, K. Stahl, P. Worden, W. Bencze, S. Buchman, B. Clarke, A. Al-Jadaan, H. Al-Jibreen, J. Li, J. Lipa, J. Lockhart, B. Al-Suwaidan, M. Taber, and S. Wang, *Phys. Rev. Lett.* **106**, 221101 (2011).

[2] C. Chakraborty and P. Pradhan, *EPJ C* **73**, 2536 (2013), [10.1140/epjc/s10052-013-2536-1](https://doi.org/10.1140/epjc/s10052-013-2536-1).  
[3] C. Chakraborty and P. Majumdar, *Classical and Quantum Gravity* **31**, 075006 (2014).  
[4] A. Das, *J. Math. Phys. (N.Y.)* **20**, 740 (1979).  
[5] W. Bonnor, *Phys. Lett. A* **158**, 23 (1991).  
[6] L. Herrera, G. A. Gonzalez, L. A. Pachón, and J. A. Rueda, *Class. Quantum Grav.* **23**, 2395 (2006), [arXiv:gr-qc/0605087](https://arxiv.org/abs/gr-qc/0605087).

- qc/0602040.
- [7] V. S. Manko, E. D. Rodchenko, B. I. Sadovnikov, and J. Sod-Hoffs, *Class. Quantum Grav.* **23**, 5389 (2006), gr-qc/0604091.
  - [8] L. Herrera and W. Barreto, *Phys. Rev. D* **86**, 064014 (2012), arXiv:1206.0413 [gr-qc].
  - [9] L. Herrera, *Gen. Relat. Gravit.* **46**, 1654 (2013), 10.1007/s10714-013-1654-2.
  - [10] M. S. Bhatia, S. Bonazzola, and G. Szamosi, *Astron. Astrophys.* **3**, 206 (1969).
  - [11] P. Ghosh and F. K. Lamb, *Astrophys. J. Lett.* **223**, L83 (1978).
  - [12] P. Ghosh and F. K. Lamb, *Astrophys. J.* **322**, 259 (1979).
  - [13] M. Camenzind, in *Reviews in Modern Astronomy*, Reviews in Modern Astronomy, edited by G. Klare, pp. 234–265.
  - [14] A. Koenigl, *Astrophys. J. Lett.* **370**, L39 (1991).
  - [15] M. M. Romanova, G. V. Ustyugova, A. V. Koldoba, and R. V. E. Lovelace, *Astrophys. J.* **578**, 420 (2002), arXiv:astro-ph/0209426.
  - [16] M. M. Romanova, G. V. Ustyugova, A. V. Koldoba, J. V. Wick, and R. V. E. Lovelace, *Astrophys. J.* **595**, 1009 (2003), arXiv:astro-ph/0401375.
  - [17] M. M. Romanova, G. V. Ustyugova, A. V. Koldoba, and R. V. E. Lovelace, *Astrophys. J.* **610**, 920 (2004), arXiv:astro-ph/0404496.
  - [18] M. Long, M. M. Romanova, and R. V. E. Lovelace, *Astrophys. J.* **634**, 1214 (2005), arXiv:astro-ph/0510659.
  - [19] W. Becker, ed., *Neutron Stars and Pulsars*, Reviews in Modern Astronomy, Vol. 357 (Springer Berlin Heidelberg, 2009).
  - [20] . O. Güver, Göğüş, *Mon. Not. Roy. Astr. Soc.* **418**, 2773 (2011).
  - [21] S. V. Berdyugina, *Living Reviews in Solar Physics* **2**, 20 (2005).
  - [22] J.-F. Donati and A. Collier Cameron, *Mon. Not. Roy. Astr. Soc.* **291**, 1 (1997).
  - [23] J.-F. Donati, A. Collier Cameron, G. A. J. Hussain, and M. Semel, *Mon. Not. Roy. Astr. Soc.* **302**, 437 (1999).
  - [24] M. Jardine, K. Wood, A. Collier Cameron, J.-F. Donati, and D. H. Mackay, *Mon. Not. Roy. Astr. Soc.* **336**, 1364 (2002), arXiv:astro-ph/0207522.
  - [25] M. Jardine, A. C. Cameron, J.-F. Donati, S. G. Gregory, and K. Wood, *Mon. Not. Roy. Astr. Soc.* **367**, 917 (2006).
  - [26] C. P. Johnstone, M. Jardine, S. G. Gregory, J.-F. Donati, and G. Hussain, *Monthly Notices of the Royal Astronomical Society* **437**, 3202 (2014).
  - [27] I. Appenzeller and R. Mundt, *Astron. Astrophys. Rev.* **1**, 291 (1989).
  - [28] J.-F. Donati, M. M. Jardine, S. G. Gregory, P. Petit, J. Bouvier, C. Dougados, F. Ménard, A. Collier Cameron, T. J. Harries, S. V. Jeffers, and F. Paltou, *Mon. Not. Roy. Astr. Soc.* **380**, 1297 (2007), arXiv:0709.1414.
  - [29] G. D. Schmidt, H. C. Harris, J. Liebert, D. J. Eisenstein, S. F. Anderson, J. Brinkmann, P. B. Hall, M. Harvanek, S. Hawley, S. J. Kleinman, G. R. Knapp, J. Krzesinski, D. Q. Lamb, D. Long, J. A. Munn, E. H. Neilsen, P. R. Newman, A. Nitta, D. J. Schlegel, D. P. Schneider, N. M. Silvestri, J. A. Smith, S. A. Snedden, P. Szkody, and D. Vanden Berk, *Astrophys. J.* **595**, 1101 (2003), arXiv:astro-ph/0307121.
  - [30] K. M. Vanlandingham, G. D. Schmidt, D. J. Eisenstein, H. C. Harris, S. F. Anderson, P. B. Hall, J. Liebert, D. P. Schneider, N. M. Silvestri, G. S. Stinson, and M. A. Wolfe, *Astrophys. J.* **130**, 734 (2005).
  - [31] S. J. Kleinman, S. O. Kepler, D. Koester, I. Pelisoli, V. Peanha, A. Nitta, J. E. S. Costa, J. Krzesinski, P. Dufour, F.-R. Lachapelle, P. Bergeron, C.-W. Yip, H. C. Harris, D. J. Eisenstein, L. Althaus, and A. Crisico, *Astrophys. J. Suppl. S.* **204**, 5 (2013).
  - [32] K. Beuermann, F. Euchner, K. Reinsch, S. Jordan, and B. T. Gänsicke, *A&A* **463**, 647 (2007).
  - [33] S. Jordan, R. Aznar Cuadrado, R. Napiwotzki, H. M. Schmid, and S. K. Solanki, *A&A* **462**, 1097 (2007).
  - [34] F. Euchner, S. Jordan, K. Reinsch, K. Beuermann, and B. T. Gänsicke, in *14th European Workshop on White Dwarfs*, Astronomical Society of the Pacific Conference Series, Vol. 334, edited by D. Koester and S. Moehler (Astronomical Society of the Pacific, San Francisco, 2005) p. 269, arXiv:astro-ph/0410199.
  - [35] G. Landstreet, J. D.; Mathys, *Astron. Astrophys.* **359**, 213 (2000).
  - [36] S. Olausen and M. Kaspi, (2014), <http://arxiv.org/abs/1309.4167>.
  - [37] S. Mereghetti, *Astron. Astrophys. Rev.* **15**, 225 (2008).
  - [38] N. R. et. al., *Astrophys. J.* **770**, L73 (2013).
  - [39] T. Güver, E. Göğüş, and F. Özel, *Mon. Not. Roy. Astr. Soc.* **418**, 2773 (2011).
  - [40] A. Gutierrez-Ruiz, L. A. Pachon, and C. Valenzuela-Toledo, *Universitas Scientiarum* **19**, 63.
  - [41] L. A. Pachón, J. A. Rueda, and C. A. Valenzuela-Toledo, *Astrophys. J.* **756**, 82 (2012), arXiv:1112.1712 [astro-ph.SR].
  - [42] L. A. Pachón, J. A. Rueda, and J. D. Sanabria-Gómez, *Phys. Rev. D* **73**, 104038 (2006), arXiv:gr-qc/0606060.
  - [43] D. Bini, A. Gericco, O. Luongo, and H. Quevedo, *Classical. Quant. Grav.* **26**, 225006 (2009), arXiv:0909.4150 [gr-qc].
  - [44] T. Johannsen and D. Psaltis, *Phys. Rev. D* **83**, 124015 (2011), arXiv:1105.3191 [gr-qc].
  - [45] S. Toktarbay and H. Quevedo, *Gravitation Cosmol.* **20**, 252 (2014).
  - [46] F. J. Ernst, *Phys. Rev.* **172**, 1850 (1968).
  - [47] N. R. Sibgatullin, *Oscillations and Waves in Strong Gravitational and Electromagnetic Fields, XIV*, 362 pp. 26 figs.. Springer-Verlag Berlin Heidelberg New York. Also *Texts and Monographs in Physics* (1991).
  - [48] V. S. Manko and N. R. Sibgatullin, *Class. Quantum Grav.* **10**, 1383 (1993).
  - [49] L. A. Pachón and J. D. Sanabria-Gómez, *Class. Quantum Grav.* **23**, 3251 (2006), arXiv:gr-qc/0603081.
  - [50] F. J. Ernst, V. S. Manko, and E. Ruiz, *Class. Quant. Grav.* **23**, 4945 (2006), arXiv:gr-qc/0701112 [gr-qc].
  - [51] L. Stella and M. Vietri, *Phys. Rev. Lett.* **82**, 17 (1999), arXiv:astro-ph/9812124.
  - [52] S. M. Morsink and L. Stella, *Astrophys. J.* **513**, 827 (1999), arXiv:astro-ph/9808227.
  - [53] L. Stella and M. Vietri, *Astrophys. J. Lett.* **492**, L59 (1998), arXiv:astro-ph/9709085.
  - [54] E. Ruiz, V. S. Manko, and J. Martín, *Phys. Rev. D* **51**, 4192 (1995).
  - [55] A. Papapetrou, *Annalen der Physik, (Berlin)/(N.Y.)/(Paris)* **447**, 309 (1953).
  - [56] C. Hoenselaers and Z. Perjés, *Class. Quantum Grav.* **7**, 1819 (1990).

- [57] T. P. Sotiriou and T. A. Apostolatos, *Class. Quant. Grav.* **21**, 5727 (2004).
- [58] B. Werner, *Neutron Stars and Pulsars* (2009).
- [59] M. Bocquet, S. Bonazzola, E. Gourgoulhon, and J. Novak, *Astron. Astrophys.* **301**, 757 (1995), [gr-qc/9503044](#).
- [60] F. L. Dubeibe, L. A. Pachón, and J. D. Sanabria-Gómez, *Phys. Rev. D* **75**, 023008 (2007), [arXiv:gr-qc/0701065](#).
- [61] R. M. Wald, *General relativity* (Chicago, University of Chicago Press, 1984, 504 p., 1984).
- [62] G. Pappas and T. A. Apostolatos, *Phys. Rev. Lett.* **108**, 231104 (2012), [arXiv:1201.6067 \[gr-qc\]](#).
- [63] V. S. Manko, E. Rodchenko, B. Sadovnikov, and J. Sod-Hoffs, *Class. Quant. Grav.* **23**, 5389 (2006).
- [64] L. Rezzolla, B. J. Ahmedov, and J. C. Miller, *MNRAS* **322**, 723 (2001).
- [65] O. Karas, V. Kopáček and D. Kunneriath, *Classical and Quantum Gravity* **29**, 035010 (2012).
- [66] O. Karas, V. Kopáček and D. Kunneriath, *Acta Polytechnica* **54**, 398 (2014).
- [67] M. van der Klis, *Adv. Space Res.* **38**, 2675 (2006).
- [68] J. Homan, *Astrophys. J. Lett.* **760**, L30 (2012).
- [69] F. D. Ryan, *Phys. Rev. D* **52**, 5707 (1995).
- [70] J. M. Bardeen, W. H. Press, and S. A. Teukolsky, *Astrophys. J.* **178**, 347 (1972).
- [71] M. Stute and M. Camenzind, *Mon. Not. Roy. Astr. Soc.* **336**, 831 (2002), [arXiv:astro-ph/0301466](#).
- [72] J.-F. Donati, I. D. Howarth, M. M. Jardine, P. Petit, C. Catala, J. D. Landstreet, J.-C. Bouret, E. Alecian, J. R. Barnes, T. Forveille, F. Paletou, and N. Manset, *Mon. Not. Roy. Astr. Soc.* **370**, 629 (2006), [arXiv:astro-ph/0606156](#).
- [73] A. Mastrano, P. D. Lasky, and A. Melatos, *Mon. Not. Roy. Astr. Soc.* **434**, 1658 (2013).
- [74] B. Haskell, R. Ciolfi, F. Pannarale, and L. Rezzolla, *Mon. Not. Roy. Astr. Soc.* **438**, L71 (2014), [arXiv:1309.3885 \[astro-ph.SR\]](#).
- [75] A. Gutierrez-Ruiz, C. Valenzuela-Toledo, and L. A. Pachon, (to be published) (2015).
- [76] N. Bretón, V. S. Manko, and J. Aguilar Sánchez, *Class. Quantum Grav.* **16**, 3725 (1999).
- [77] W. Kinnersley, *J. Math. Phys.(N.Y.)* **18**, 1529 (1977).

## Appendix A: Metric Functions of the Developed Solution

This appendix summarizes the relevant equations of the developed metric. The potentials in the symmetry axis can be written as [54]:

$$e(z) = 1 + \sum_{i=3}^3 \frac{e_i}{z - \beta_i}, \quad f(z) = \sum_{i=3}^3 \frac{f_i}{z - \beta_i}, \quad (\text{A1})$$

with

$$e_j = (-1)^j \frac{2m\beta_j^2}{(\beta_j - \beta_k)(\beta_j - \beta_i)}, \quad f_j = \frac{i\zeta + (\varsigma + i\mu)\beta_j}{(\beta_j - \beta_k)(\beta_j - \beta_i)}, \quad i, k \neq j. \quad (\text{A2})$$

Then, the Ernst potentials and the metric functions in whole spacetime are derived with the aid of the Sibgatullin's integral method [47, 48]. By using the representation proposed in Ref. [76] and used also Ref. [42],

$$\mathcal{E} = \frac{A+B}{A-B}, \quad \Phi = \frac{C}{A-B}, \quad (\text{A3})$$

$$f = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A-B)(\bar{A}-\bar{B})}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{K\bar{K} \prod_{n=1}^6 r_n}, \quad \omega = \frac{\text{Im}[(A+B)\bar{H} - (\bar{A}+\bar{B})G - C\bar{I}]}{A\bar{A} - B\bar{B} + C\bar{C}},$$

where

$$\begin{aligned}
A &= \sum_{1 \leq i < j < k \leq 6} a_{ij\,k} r_i r_j r_k, \quad B = \sum_{1 \leq i < j \leq 6} b_{ij} r_i r_j, \quad C = \sum_{1 \leq i < j \leq 6} c_{ij} r_i r_j, \quad K = \sum_{1 \leq i < j < k \leq 6} a_{ij\,k}, \\
H &= z A - (\beta_1 + \beta_2 + \beta_3) B + \sum_{1 \leq i < j < k \leq 6} h_{ij\,k} r_i r_j r_k + \sum_{1 \leq i < j \leq 6} (\alpha_i + \alpha_j) b_{ij} r_i r_j, \\
G &= -(\beta_1 + \beta_2 + \beta_3) A + z B + \sum_{1 \leq i < j \leq 6} g_{ij} r_i r_j + \sum_{1 \leq i < j < k \leq 6} (\alpha_i + \alpha_j + \alpha_k) a_{ij\,k} r_i r_j r_k, \\
I &= (f_1 + f_2 + f_3)(A - B) + (\beta_1 + \beta_2 + \beta_3 - z) C + \sum_{1 \leq i < j < k \leq 6} p_{ij\,k} r_i r_j r_k + \sum_{i=1}^6 p_i r_i \\
&+ \sum_{1 \leq i < j \leq 6} [p_{ij} - (\alpha_i + \alpha_j) c_{ij}] r_i r_j,
\end{aligned}$$

with

$$\begin{aligned}
r_i &= \sqrt{\rho^2 + (z - \alpha_i)^2}, \quad a_{ij\,k} = (-1)^{i+j+1} \Lambda_{ijk} \Gamma_{l|mn}, \quad b_{ij} = (-1)^{i+j} \lambda_{ij} H_{l|mnp}, \\
c_{ij} &= (-1)^{i+j} \lambda_{ij} [f(\alpha_i) \Gamma_{m|np} - f(\alpha_m) \Gamma_{n|pl} + f(\alpha_n) \Gamma_{p|lm} - f(\alpha_p) \Gamma_{l|mn}], \\
h_{ij\,k} &= (-1)^{i+j+k} \Lambda_{ijk} (e_1^* \delta_{23|lmn} + e_2^* \delta_{31|lmn} + e_3^* \delta_{12|lmn}), \\
g_{ij} &= (-1)^{i+j} \lambda_{ij} (\alpha_l \Gamma_{m|np} - \alpha_m \Gamma_{n|pl} + \alpha_n \Gamma_{p|lm} - \alpha_p \Gamma_{l|mn}), \\
p_i &= (-1)^i D_i [f(\alpha_i) H_{m|nps} - f(\alpha_m) H_{n|psl} + f(\alpha_n) H_{p|slm} - f(\alpha_p) H_{s|lmn} + f(\alpha_s) H_{l|mnp}], \\
p_{ij} &= (-1)^{i+j} \lambda_{ij} (e_1^* \Upsilon_{23|lmnp} + e_2^* \Upsilon_{31|lmnp} + e_3^* \Upsilon_{12|lmnp}), \\
p_{ij\,k} &= (-1)^{i+j+1} \Lambda_{ijk} (e_1^* \Psi_{23|lmn} + e_2^* \Psi_{31|lmn} + e_3^* \Psi_{12|lmn}),
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{ij} &= (\alpha_i - \alpha_j) D_i D_j, \quad \Lambda_{ijk} = (\alpha_i - \alpha_j)(\alpha_i - \alpha_k)(\alpha_j - \alpha_k) D_i D_j D_k, \quad D_i = \frac{1}{(\alpha_i - \beta_1)(\alpha_i - \beta_2)(\alpha_i - \beta_3)}, \\
\Gamma_{l|mn} &= H_3(\alpha_l) \Delta_{12|mn} + H_3(\alpha_m) \Delta_{12|nl} + H_3(\alpha_n) \Delta_{12|lm}, \quad \Delta_{lm|np} = H_l(\alpha_n) H_m(\alpha_p) - H_l(\alpha_p) H_m(\alpha_n), \\
H_l(\alpha_n) &= \frac{2 \prod_{p \neq n} (\alpha_p - \beta_l^*)}{\prod_{k \neq l}^3 (\beta_l^* - \beta_k^*) \prod_{k=1}^3 (\beta_l^* - \beta_k)} - 2 \sum_{k=1}^3 \frac{f_l^* f_k}{(\beta_l^* - \beta_k)(\alpha_n - \beta_k)}, \quad \delta_{lm|nps} = \Delta_{lm|np} + \Delta_{lm|ps} + \Delta_{lm|sn}, \\
h_{l|mnp} &= H_3(\alpha_l) \delta_{12|mnp}, \quad H_{l|mnp} = h_{l|mnp} + h_{m|npl} + h_{n|plm} + h_{p|lmn}, \\
\Psi_{lm|nps} &= f(\alpha_n) \Delta_{lm|ps} + f(\alpha_p) \Delta_{lm|sn} + f(\alpha_s) \Delta_{lm|np}, \\
\Upsilon_{lm|nprs} &= f(\alpha_n) \delta_{lm|prs} - f(\alpha_p) \delta_{lm|rsn} + f(\alpha_r) \delta_{lm|snp} - f(\alpha_s) \delta_{lm|npr},
\end{aligned}$$

being  $\alpha$ 's the roots of the Sibgatullin's equation [47, 48]

$$e(z) + \tilde{e}(z) + 2\tilde{f}(z)f(z) = 0. \quad (\text{A4})$$

## Appendix B: Electromagnetic Field: Analytic Form

The  $A_t$  potential is the real part of the electromagnetic Ernst potential  $\Phi$ , and the potential  $A_\phi$  can be calculated as the real part of the Kinnersley potential  $\mathcal{K} = A_\phi + iA'_t$  [77], which can be obtained using the Sibgatullin's method and can be written as

$$\mathcal{K} = -i \frac{I(f_1 + f_2)}{A - B}. \quad (\text{B1})$$



Thus, the closed-form expressions for the electric and magnetic fields are

$$E_\rho = -\frac{\Lambda}{\sqrt{|A|^2 - |B|^2 + |C|^2}} \operatorname{Re} \left\{ \left[ \frac{C_{,\rho} - C \ln(A - B)_{,\rho}}{A - B} \right] \right\},$$

$$E_z = -\frac{\Lambda}{\sqrt{|A|^2 - |B|^2 + |C|^2}} \operatorname{Re} \left\{ \left[ \frac{C_{,z} - C \ln(A - B)_{,z}}{A - B} \right] \right\},$$

$$B_\rho = \frac{\operatorname{Im}(A - B)\bar{H} + (\bar{A} - \bar{B})G - C\bar{I}}{\rho|A - B|^2} E_z + \frac{\Lambda\sqrt{|A|^2 - |B|^2 + |C|^2}}{\rho|A - B|^2} \operatorname{Im} \left\{ \frac{(\bar{f}_1 + \bar{f}_2) [\bar{I}_{,z} - \bar{I} \ln(\bar{A} - \bar{B})_{,z}]}{A - B} \right\},$$

$$B_z = -\frac{\operatorname{Im}(A - B)\bar{H} + (\bar{A} - \bar{B})G - C\bar{I}}{\rho|A - B|^2} E_\rho + \frac{\Lambda\sqrt{|A|^2 - |B|^2 + |C|^2}}{\rho|A - B|^2} \operatorname{Im} \left\{ \frac{(\bar{f}_1 + \bar{f}_2) [\bar{I}_{,\rho} - \bar{I} \ln(\bar{A} - \bar{B})_{,\rho}]}{A - B} \right\}.$$

when

$$\Lambda = \frac{|K|^2 \prod_{n=1}^6 r_n}{|A - B|}.$$


---